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Extremal Permutations with Respect to Weak Majorizations

F. K. Hwang

Let $Z = (z_1, z_2, \ldots, z_n)$ denote a permutation of an n-set. Define

$$F_l(Z) = \{f(z_1, z_2), \ldots, f(z_{n-1}, z_n)\}$$

and its cyclic version

$$F_c(Z) = \{f(z_1, z_2), \ldots, f(z_{n-1}, z_n), f(z_n, z_1)\},$$

where $f(x, y)$ increases in $\max\{x, y\}$ and decreases in $\min\{x, y\}$. We give conditions on f such that extremal permutations with respect to weak majorization can be found. We then use the weak majorization property to obtain extremal permutations for

$$L_f(Z) = g(f(z_1, z_2), \ldots, f(z_{n-1}, z_n))$$

and

$$C_f(Z) = g(f(z_1, z_2), \ldots, f(z_{n-1}, z_n), f(z_n, z_1))$$

when g is either convex increasing or concave decreasing.

1. Introduction

Related to the sorting problem in computer science, several measures of various types of the disorderedness of the input data have been proposed in the literature. One such measure, called the *oscillation* of the input data $Z = (z_1, \ldots, z_n)$ by Leveopoulos and Petersson [7], is defined as

$$O_{sc}(Z) = \sum_{i=1}^{n-1} (|z_i - z_{i+1}| - 1).$$

Motivated by this, Chao and Liang [1] considered

$$L_h(Z) = \sum_{i=1}^{n-1} h(|z_i - z_{i+1}|)$$

and

$$C_h(Z) = \sum_{i=1}^{n} h(|z_i - z_{i+1}|),$$

where h is increasing. We will refer to a permutation in $L_h(z)$ as a *linear permutation* and a permutation in $C_h(z)$ as a *cyclic permutation.* A permutation which minimizes or maximizes $L_h(z)(C_h(z))$ is called a *minimum* or a *maximum linear* (*cyclic*) *permutation.* Chao and Liang obtained:

(i) the minimum linear permutation and the minimum cyclic permutation;
(ii) the maximum linear permutation and the maximum cyclic permutation when h is convex;
(iii) the maximum cyclic permutation and a candidate set for the maximum linear permutation when h is concave (but see comments on Results 4 and 5 in Section 2).

The minimum linear permutation problem has been studied in [4, 9] in the context of the variation-reducing properties of decreasing rearrangement. Let $(z_{[1]} \geqslant z_{[2]} \geqslant \cdots \geqslant z_{[n]})$

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denote the ordered set of Z; then the decreasing rearrangement is the permutation $l_n\downarrow = (z_{[1]}, z_{[2]}, \ldots, z_{[n]})$. By treating a permutation and its reverse as the same, Chong [2] proved that $l_n\downarrow$ weakly submajorizes all other permutations Z and hence that $l_n\downarrow$ is the minimum linear permutation if h is convex increasing.

In this paper we show that the extremal permutations obtained by Chao and Liang are extremal under a much broader setting, i.e. they are extremal permutations with respect to weak submajorization and weak supermajorization. We also make the following generalizations:
(i) replace the increasing function $h(|x-y|)$ by $f(x, y)$ which increases in $\max\{x, y\}$ and decreases in $\min\{x, y\}$;
(ii) replace the convexity and concavity conditions on h by the L-subadditive and L-superadditive conditions on f;
(iii) replace the summation function in L_h and L_c by a Schur convex increasing or a Schur concave increasing function.

We will verify that (ii) is indeed a generalization in Section 4 after the L-additivity is defined. From (i) and (ii), we have

$$L_f(Z) = g(f(z_1, z_2), \ldots, f(z_{n-1}, z_n))$$

and

$$C_f(Z) = g(f(z_1, z_2), \ldots, f(z_{n-1}, z_n), f(z_n, z_1)).$$

2. Background

An n-set $X = (x_1, x_2, \ldots, x_n)$ is said [8] to *weakly submajorize* another n-set $Y = (y_1, y_2, \ldots, y_n)$, denoted by $Y \prec_w X$, if

$$\sum_{i=1}^{k} y_{[i]} \leqslant \sum_{i=1}^{k} x_{[i]} \qquad \text{for all } k = 1, \ldots, n,$$

and to *weakly supermajorize* Y, denoted by $Y \prec^w X$ if

$$\sum_{i=k}^{n} y_{[i]} \geqslant \sum_{i=k}^{n} x_{[i]} \qquad \text{for all } k = 1, \ldots, n.$$

It is well known that weak majorization is additive, i.e. if $A\, {}_w\!\succ B$ and $C\, {}_w\!\succ D$, then $A \cup C\, {}_w\!\succ B \cup D$. The same goes for ${}^w\!\succ$. Consider a permutation $Z = (z_1, z_2, \ldots, z_n)$ of a given n-set and define

$$F_l(Z) = \{f(z_1, z_2), \ldots, f(z_{n-1}, z_n)\}$$

and its cyclic version

$$F_c(Z)) = \{f(z_1, z_2), \ldots, f(z_{n-1}, z_n), f(z_n, z_1)\}.$$

For easier presentation, we will now assume that Z and f are such that all inequalities encountered in the analysis are strict. The non-strict case then follows by a continuity argument, while a unique solution may lose its uniqueness.

We will view each z_i as a *point* and each pair of consecutive points an *edge* (in the F_c case z_n and z_1 are considered consecutive). In particular, edge i refers to the pair (z_i, z_{i+1}). Two edges are *adjacent* if they share a common point and *non-adjacent* if otherwise. Note that edge i is associated with the interval $[z_i, z_{i+1}]$. Let edge i and edge j be non-adjacent. The pair (i, j) is called *disjoint* if their corresponding intervals do not overlap, *nested* if one interval contains the other and *crossing* if otherwise. The notions

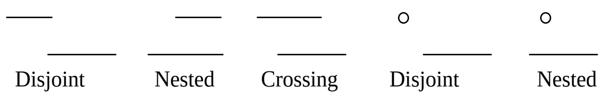

FIGURE 1. Names for pairs of edges.

of disjointness and nestedness are extended to the case in which either edge i or edge j is *degenerate,* i.e. it consists of the single point z_1 or z_n (see Figure 1).

There are other names attached to the pair (i, j) depending on other relations. Du and Hwang [3] called the pair *singular* if $(z_j - z_{i+1})(z_{j+1} - z_i) > 0$ and *non-singular* if otherwise. Note that a disjoint, non-degenerate pair is always singular. We also call the pair (i, j) *synchronous* if $(z_{j+1} - z_j)(z_{i+1} - z_i) > 0$ and *asychronous* if otherwise. Although not using these names, the notion of synchronism was introduced by Chao and Liang. In particular, for two adjacent edges $\{z_{i-1}, z_i)$ and (z_i, z_{i+1}), they called the point z_i:

(i) a *rise* if $(i-1, i)$ is synchronous and $z_i > z_{i-1}$;
(ii) a *fall* if $(i-1, i)$ is asynchronous and $z_i < z_{i-1}$;
(iii) a *peak* if $z_i > \max\{z_{i-1}, z_{i+1}\}$ (z_i or z_n a *half peak* if $z_1 > z_2$ or $z_n > z_{n-1}$);
(iv) a *valley* if $z_i < \min\{z_{i-1}, z_{i+1}\}$ (z_1 or z_n a *half valley* if $z_1 < z_2$ or $z_n < z_{n-1}$).

Clearly, if z_i is a peak or a valley, then $(i-1, i)$ is asynchronous.

A degenerate edge z_1 (or z_n) is asynchronous disjoint with edge i if z_1, z_i, z_{i+1} (or z_i, z_{i+1}, z_n) is monotone.

Chao and Liang proved the following results which we will use later (in giving a sequence, we represent $z_{[i]}$ by $[i]$).

RESULT 1. The unique cyclic permutation with no singular pair is $\bar{c}_n$ as shown in Figure 2 (this result was proved earlier by Du and Hwang).

RESULT 2. There are at most two linear permutations $\bar{l}_n(\lfloor b/2 \rfloor)$ and $\bar{l}_n(\lceil n/2 \rceil)$ with no singular pair, rise or fall, which can be obtained from $\bar{c}_n$ by cutting either the edge $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1)$ or the edge $(\lceil n/2 \rceil, \lceil n/2 \rceil + 1)$. Note that $\bar{l}_n(\lfloor n/2 \rfloor)$ and $\bar{l}_n(\lceil n/2 \rceil)$ are the same if n is even, and differ only in one edge if n is odd.

RESULT 3. The unique cyclic permutation with no non-singular pair is $\underline{c}_n$ as shown in Figure 3.

RESULT 4. Let C_n denote the set of cyclic permutations with no synchronous disjoint

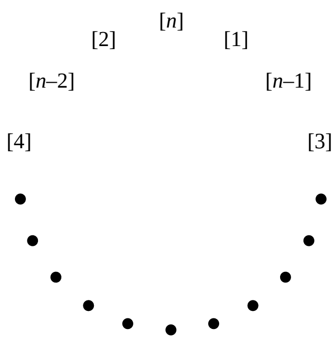

FIGURE 2. The cyclic permutation $\bar{c}_n$.

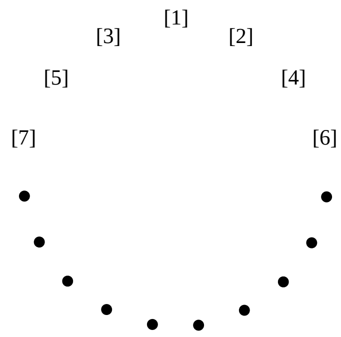

FIGURE 3. The cyclic permutation c_n.

pair or asynchronous nested pair, with 0 or 2 rise and fall for even n and with one rise and fall for odd n. Then C_n consists of a unique member, as shown in Figure 4.

Chao and Liang obtained the $n = 2k$ case by arguing that a member of C_n cannot have both a rise and a fall. However, this claim is wrong. For example, the permutation ([2], [5], [8], [3], [7], [4], [1], [6]) is in C_8, while the permutations ([2], [6], [10], [4], [8], [3], [9], [5], [1], [7]) and ([2], [6], [10], [4], [9], [5], [1], [7], [3], [8]) are in C_{10}. In fact, for every even $n = 2k \geqslant 8$, there are members in C_n with one rise and one fall.

RESULT 5. Let L_n denote the set of linear permutations with no synchronous disjoint pair or asynchronous nested pair, with no rise and fall for even n, and with one rise and fall for odd n. Then L_{2k} consists of a single member which can be obtained from the single member of C_{2k+1} by deleting point $[2k + 1]$ and its two edges. Chao and Liang also gave L_{2k+1}, but some care has to be taken since one permutation in L_9 starts and ends with the same subsequence ([6], [2], [8], [4]), while another permutation in L_{13} starts with ([8], [2], [10], [4]) and ends with ([10], [4], [12], [6]).

RESULT 6. The unique cyclic permutation with no crossing pair is $c_n\downarrow = ([1], [2], [3], \ldots, [n])$.

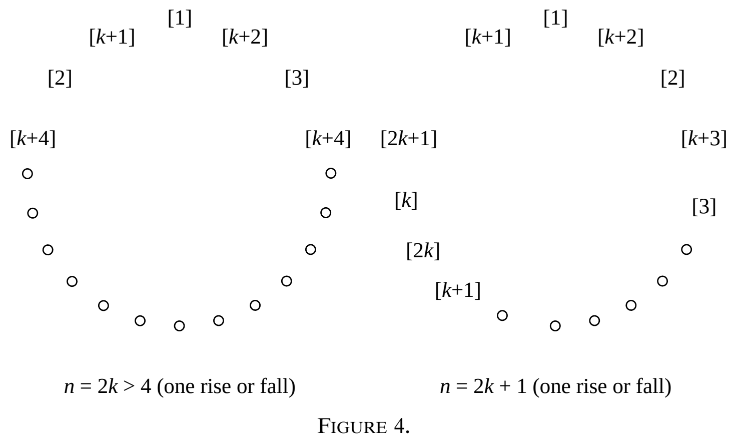

FIGURE 4.

3. The General Distance Function

A function $f(x, y)$ was called a *general distance function* (*gdf*) in [6] if f is symmetric and $f(x, y)$ increases in $\max\{x, y\}$ and decreases in $\min\{x, y\}$. The usual distance function $d(x, y) = |x - y|$ certainly is a *gdf*. It is straightforward to verify the following.

Lemma 3.1. *Given four points* $u > v > x > y$, *then* $\{f(u, x), f(v, y)\}_w > \{f(u, v), f(x, y)\}$ *and* $\{f(u, x), f(v, y)\} <^w \{f(u, v), f(x, y)\}$.

Lemma 3.1 says that a crossing pair submajorizes and is supermajorized by a disjoint pair.

For a given *gdf* $f(x, y)$, a permutation is *super-maximal* or *super-minimal* (*sub-maximal* or *sub-mimimal*) if it is a maximal or minimal element in the partial order $<^w$ ($<_w$).

Lemma 3.2. *There is no synchronous disjoint pair in a super-minimal permutation.*

Proof. Let Z be a permutation containing a synchronous disjoint pair (i, j). Without loss of generality, assume that $i < j$. Let Z' be the permutation obtained from Z by reversing the subsequence $(z_{i+1}, z_{i+2}, \ldots, z_j)$. Then $F(Z')$ differs from $F(Z)$ by replacing $f(z_i, z_{i+1})$ with $f(z_i, z_j)$ and $f(z_j, z_{j+1})$ with $f(z_{i+1}, z_{j+1})$, i.e. replacing two disjoint edges by two crossing edges. By Lemma 3.1, $F(Z')_w > F(Z)$ and $F(Z') <^w F(Z)$. The second half of Lemma 3.2 can be proved similarly. □

Corollary 3.3. *Consider a super-minimal permutation. If* z_i *is a rise, then* $z_1 > z_i > z_n$, z_1 *is a half peak and* z_n *a half valley. If* z_i *is a fall, then* $z_1 < z_i < z_n$, z_1 *is a half valley and* z_n *a half peak.*

Proof. If $z_1 < z_i$, then z_1 and (z_i, z_{i+1}) from a synchronous disjoint pair. If $z_1 > z_i$ but is a half valley, then $(1, i - 1)$ is a synchronous disjoint pair. The other cases can be proved similarly. □

Lemma 3.4. *There exist at most one rise and one fall in a super-miminal permutation. The rise and the fall cannot coexist if the permutation is linear.*

Proof. Suppose that there exist two rises z_i and z_j. Without loss of generality, assume $z_i < z_j$. Then $(i - 1, j)$ is a synchronous disjoint pair, contradicting Lemma 3.2. An analogous argument proves the non-existence of two falls. Finally, suppose that there exist a rise z_i and a fall z_j in a linear permutation. Then $z_i < z_1 < z_j$ and $z_j < z_n < z_1$ by Corollary 3.3, an absurdity. □

Lemma 3.5. *The number of rises and falls in a super-minimal cyclic permutation is* 0 *or* 2 *if n is even, and is* 1 *if n is odd. A maximal linear permutation has no rise and fall if n is even.*

Proof. If we ignore rises and falls in a permutation, then peaks and valleys alternate. Hence a cyclic permutation has as many peaks as valleys.

By Lemma 3.4, a sub-maximal linear permutation has at most one rise and fall.

Suppose that it has one. By Corollary 3.3, one of z_1 and z_n is a half peak and the other a half valley. Since peaks and valleys, including those half ones, alternate, their numbers must be equal.

In either case, the numbers of peaks and valleys are equal. Hence the number of rises and falls has the same parity as n. Lemma 3.5 now follows from Lemma 3.4. □

While the above properties are useful in restricting maximal permutations, additional assumptions have to be introduced in the next section to pin these permutations down. However, the minimal linear permutation can be obtained without any additional assumption.

THEOREM 3.6. *$l_n\downarrow$ is the minimal linear permutation.*

PROOF. We prove Theorem 3.6 by induction on n. The $n=2$ case is trivially true. Let Z be a minimal permutation where $z_{[n]}$ lies between $z_{[i]}$ and $z_{[j]}$. Let Z' be obtained from Z by deleting z_n. Then

$$F(Z) = F(Z') \cup \{f(z_{[i]}, z_{[n]}), f(z_{[n]}, z_{[j]})\} \backslash f(z_{[i]}, z_{[j]})$$

and

$$F(l_n\downarrow) = F(l_{n-1}\downarrow) \cup f(z_{[n-1]}, z_{[n]}).$$

But $F(Z')\,{}_w\!\succ F(l_{n-1}\downarrow)$ by induction, and

$$f(z_{[i]}, z_{[n]}) \geqslant f(z_{[n-1]}, z_{[n]}), \qquad f(z_{[n]}, z_{[j]}) > f(z_{[i]}, z_{[j]}).$$

Hence

$$F(Z)\,{}_w\!\succ F(l_n\downarrow).$$

Next let $Z \neq Z_n\downarrow$ be a minimal permutation with z_n as an endpoint. Let Z' be obtained from Z by deleting $z_{[n]}$. Assume that $z_{[i]}$ follows $z_{[n]}$ in Z. Then

$$F(Z) = F(Z') \cup f(z_{[n]}, z_{[i]}), \qquad F(l_n\downarrow) = F(l_{n-1}\downarrow) \cup f(z_{[n]}, z_{[n-1]}).$$

Again,

$$F(Z')\,{}_w\!\succ F(l_{n-1}\downarrow)$$

by induction, and

$$f(z_{[n]}, z_{[i]}) \geqslant f(z_{[n]}, z_{[n-1]}).$$

Hence

$$F(Z)\,{}_w\!\succ F(l_n\downarrow).$$

The proof for ${}^w\!\succ$ is similar. □

COROLLARY 3.8. *Suppose that g is Schur convex or concave increasing. Then $l_m\downarrow$ minimizes $L_f(Z)$.*

4. EXTREMAL PERMUTATIONS WHEN f IS L-SUBADDITIVE OR L-SUPERADDITIVE

A bivariate function $b(x, y)$ is *L-subadditive* [8] if $b(x, y) + b(x', y') \geqslant b(\bar{x}, \bar{y}) + b(\underline{x}, \underline{y})$, where $\bar{w} = \max\{w, w'\}$, $\underline{w} = \min\{w, w'\}$ for $w \in \{x, y\}$. It is *L-superadditive* if the above inequality is reversed.

To discuss the L-additivity property of a *gdf* $f(x, y)$, it is more convenient to transform $f(x, y)$ into $G(x, y)$, which is defined only for $x \geqslant y$, and is increasing in x and decreasing in y. Then $f(x, y)$ is L-subadditive if

$$G(x, y) + G(x', y') \geqslant G(\bar{x}, \bar{y}) + G(\underline{x}, \underline{y}),$$

and is L-superadditive if the inequality is reversed.

We now verify that generalization (ii) mentioned in Section 1 is indeed a generalization.

Lemma 4.1. *Suppose $f(x, y) = h(|x - y|)$. Then h being convex implies f is subadditive, and h being concave superadditive.*

Proof. We only prove for convex h. Let $u > v \geqslant x > y$. Then

$$\begin{aligned} f(u, y) + f(v, x) - f(u, x) - f(v, y) &= G(u, y) + G(v, x) - G(u, x) - G(v, y) \\ &= h(u - y) + h(u - x) - h(u - x) - h(v - y) \geqslant 0 \end{aligned}$$

by the convexity of h, since

$$(u - y) + (u - x) = (u - x) + (v - y), \qquad u - y \geqslant \max\{u - x, v - y\}. \qquad \square$$

It is straightforward to verify the following.

Lemma 4.2. *Suppose that f is an L-subadditive gdf. For $u > v \geqslant x > y$,*

$$\{f(u, y), f(v, x)\}\ {}_w\!\!> \{f(u, x), f(v, y)\}.$$

Lemma 4.2 says that a nested pair submajorizes a crossing pair. Combining Lemmas 3.1 and 4.2, we have the following linear order when f is an L-subadditive *gdf*:

$$\text{nested}\ {}_w\!\!> \text{crossing}\ {}_w\!\!> \text{disjoint}.$$

Lemma 4.3. *Suppose that f is an L-subadditive gdf. Then*:
(i) *a sub-maximal permutation does not contain a singular pair*;
(ii) *a sub-minimum permutation does not contain a non-singular pair.*

Proof. (i) Suppose that Z contains a singular pair (i, j). Let Z' be obtained from Z by reversing the subsequence $(z_{i+1}, z_{i+2}, \ldots, z_n)$. Then $F(Z')$ differs from $F(Z)$ only in replacing the two edges (z_i, z_{i+1}) and (z_j, z_{j+1}) with the two edges (z_i, z_j) and (z_{i+1}, z_{j+1}). There are three possibilities for the change: disjoint to crossing, disjoint to nested and crossing to nested. By Lemmas 3.1 and 4.2, $F(Z')\ {}_w\!\!> F(Z)$ regardless which change occurs. (ii) Suppose that Z contains a non-singular pair (i, j). Then reversing the same subsequence as in (i) also reverses the three changes given in (i). Hence $F(Z') <_w F(Z)$. $\square$

Theorem 4.4. *Suppose that f is an L-subadditive gdf. Then*:
(i) *$\bar{c}_n$ is the unique sub-maximal cyclic permutation*;
(ii) *$\bar{l}_n(\lfloor n/2 \rfloor)$ and $\bar{l}_n(\lceil n/2 \rceil)$ are the only two sub-maximal linear permutations*;
(iii) *$\underline{c}_n$ is the unique sub-minimal cyclic permutation*;
(iv) *$l_n\!\downarrow$ is the unique sub-minimal linear permutation.*

Proof. (i) and (iii) follow from Lemma 4.3 and Results 1 and 3 immediately. (iv) follows from Theorem 3.6. (ii) would follow from Lemma 4.3 and Result 2 if we could prove that a sub-maximal linear permutation has no rise and fall. Let z_i be a rise or fall in a linear permutation Z, and let Z' be obtained from Z by moving z_i to the end of Z. We prove that $F(Z')_w\!> F(Z)$ by proving that

$$\{f(z_{i+1}, z_{i+1}), f(z_n, z_i)\}_w\!> \{f(z_{i-1}, z_i), f(z_i, z_{i+1})\}.$$

But this is obvious since

$$\begin{aligned} f(z_{i-1}, z_{i+1}) + f(z_n, z_i) &> f(z_{i-1}, z_{i+1}) + f(z_i, z_i) \qquad \text{by } gdf, \\ &\geqslant f(z_{i-1}, z_i) + f(z_i, z_{i+1}) \qquad \text{by } L\text{-subadditivity}, \end{aligned}$$

and

$$f(z_{i-1}, z_{i+1}) \geqslant \max\{f(z_{i-1}, z_i), f(z_i, z_{i+1})\} \qquad \text{by } gdf. \qquad \square$$

COROLLARY 4.5. *Suppose that g is Schur convex increasing and f is an L-subadditive gdf. Then*:
(i) $\bar{c}_n$ *uniquely maximizes* $C_f(Z)$;
(ii) $\bar{l}_n(\lfloor n/2 \rfloor)(\bar{l}_n(\lceil n/2 \rceil))$ *uniquely maximizes* $L_f(Z)$ *if* $f(\lfloor n/2 \rfloor), \lfloor n/2 \rfloor) + 1) \geqslant (\leqslant) f(\lceil n/2 \rceil, \lceil n/2 \rceil + 1)$;
(iii) $\underline{c}_n$ *uniquely minimizes* $C_f(Z)$;
(iv) $l_n\!\downarrow$ *uniquely minimizes* $L_f(Z)$.

LEMMA 4.6. *Suppose that f is an L-superadditive gdf. For* $u > v \geqslant x > y$,

$$\{f(u, x), f(v, y)\} <^w \{f(u, y), f(v, x)\}.$$

PROOF.

$$\begin{aligned} f(u, x) + f(v, y) &= G(u, x) + G(v, y) \\ &\geqslant G(u, y) + G(v, x) = f(u, y) + f(v, x) \qquad \text{by } L\text{-superadditivity}; \end{aligned}$$

$$\min\{f(u, x), f(v, y)\} > f(v, x) \qquad \text{since } f \text{ is a } gdf. \qquad \square$$

Lemma 4.6 says that a nested pair supermajorizes a crossing pair. Combining Lemmas 3.1 and 4.6, we have the following partial order when f is an L-superadditive *gdf*:

$$\text{disjoint} \;{}^w\!\!> \text{crossing}, \qquad \text{nested} \;{}^w\!\!> \text{crossing}.$$

LEMMA 4.7. *Suppose that f is an L-superadditive gdf. Then* (*i*) *a super-minimal permutation contains neither an asynchronous nested pair n or a synchronous disjoint pair, and* (*ii*) *a super-maximal permutation does not contain a crossing pair.*

PROOF. If Z contains a forbidden pair (i, j) then reversing the subsequence $(z_{i+1}, z_{i+2}, \ldots, z_j)$ will yield a permutation Z such that $F(Z') \,{}^w\!\!> F(Z)$ by Lemmas 3.1 and 4.6. $\square$

THEOREM 4.8. *Suppose that f is an L-superadditive gdf. Then*:
(i) C_n *is the set of super-minimal cyclic permutations*;
(ii) L_n *is the set of super-minimal linear permutations*;
(iii) $c_n\!\downarrow$ *is the unique super-maximal cyclic permutation*;
(iv) $l_n\!\downarrow$ *is the unique super-maximal linear permutations.*

PROOF. (i) This follows from Lemmas 3.5 and 4.7 and Result 4: (ii) follows from

Lemmas 3.5 and 4.7 and Result 5; (iii) follows from Lemma 4.7 and Result 6; and (iv) follows from Theorem 3.6. □

COROLLARY 4.9. *Suppose that g is Schur concave decreasing and f is an L-superadditive gdf. Then*:

(i) *a member of C_n minimizes $C_f(Z)$*;
(ii) *a member of L_n minimizes $L_f(Z)$*;
(iii) *$c_n\downarrow$ uniquely maximizes $C_f(Z)$*;
(iv) *$l_n\downarrow$ uniquely maximizes $L_f(Z)$*.

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